NCsostools for SOHS decompositions of non-commutative polynomials

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(joint work with Igor Klep)

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Some notation

- $\mathbb{R}[\bar{x}] = \mathbb{R}[x_1, \ldots, x_n]$ ... algebra of real polynomials in n commutative variables $\bar{x} = (x_1, \ldots, x_n)$
- $\mathbb{R}\langle\bar{x}\rangle = \mathbb{R}\langle x_1, \ldots, x_n\rangle$ ... algebra of real polynomials in n non-commutative variables $\bar{x} = (x_1, \ldots, x_n)$ (i.e.: $x_i x_j \neq x_j x_i, \forall i, j.$)
- We equip $\mathbb{R}\langle\bar{x}\rangle$ with the **involution** $*$ which reverses words, e.g.,
$$(x_1 x_2 - x_1^2 x_3)^* = x_2 x_1 - x_3 x_1^2.$$
- We assume $x_i^* = x_i$ (symmetric variables).

- **Primal polynomial optimization problem (PPOP)**
  For a given polynomial $f(x) \in \mathbb{R}[x]$ find
  $$\boxed{\hat{f} = \inf_{x \in \mathbb{R}^n} f(x)}$$

Motivation for the commutative case

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- **Example:** Let $A = \mathbb{R}^{n \times n}$, $A^T = A$. The problem "Is $\inf_{x \in \mathbb{R}^n} \sum_{i,j} a_{ij} x_i^2 x_j^2 \geq 0$?" belongs to **co-NP**.
- Unless co-NP = NP there is no polynomial certificate for non-negativity of given polynomial.

- **Obviously:** Let $f \in \mathbb{R}[\bar{x}]$. **If exist** $q_1, \ldots, q_k \in \mathbb{R}[\bar{x}]$ such that $f(x) = \sum_i q_i(\bar{x})^2$ **then** $f \geq 0$.

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SOS decomposition

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- Polynomials q_i are **SOS** decomposition for f (certificate for non-negativity).
- $\boxed{f \text{ is SOS} \Rightarrow f \geq 0}$ **but** $\boxed{f \geq 0 \not\Rightarrow f \text{ is SOS}}$.
- **Example:** Motzkin form (1967): $M(x_1, x_2, x_3) = x_1^4 x_2^2 + x_1^2 x_2^4 + x_3^6 - 3x_1^2 x_2^2 x_3^2$ is non-negative, but **not SOS**.

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- We can say more:

  Theorem (Helton, 2002):

  $$\boxed{f \in \mathbb{R}\langle \bar{x} \rangle \text{ is SOHS} \Leftrightarrow f(x_1, \ldots, x_n) \succeq 0}$$

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- **Bessis - Moussa - Villani (BMV) conjecture, 1975**

  For symmetric matrices A, B with B positive semidefinite, the function

  $$\Phi^{A,B} : \mathbb{R} \to \mathbb{R}, t \mapsto \mathrm{tr} e^{A-tB}$$

  is the Laplace transform of a positive measure $\mu^{A,B}$ on $[0, \infty)$
- Conjecture proved for $n \leq 13$ by Klep and Schweighofer (2008) using SOHS concept.

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- **Example:** $p(x_1, x_2) = 2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 5x_2^4 \in \mathcal{F}_{2,4}$
- p is SOS because there exists a PSD matrik Q such that

$$p(x_1, x_2) = z^T Q z = (x_1^2, x_2^2, x_1 x_2) \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1 x_2 \end{bmatrix}$$

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- Semidefinite program to find Q

$$\begin{array}{rrcl} \inf & \mathrm{trace}(C^TQ) & & \\ \text{such that} & q_{1,1} & = & 2 \\ & q_{2,2} & = & 5 \\ & q_{1,3}+q_{3,1} & = & 2 \\ & q_{1,2}+q_{2,1}+q_{3,3} & = & -1 \\ & q_{2,3}+q_{3,2} & = & 0 \\ & Q & \succeq & 0. \end{array}$$

- Feasible solution Q

$$Q = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}^T \cdot \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

- **SOS decomposition:**

$$p(x_1, x_2) = \frac{1}{2}((2x_1^2 - 3x_2^2 + x_1x_2)^2 + (x_2^2 + 3x_1x_2)^2).$$

- **Main tasks:** determine vector z, construct SDP and solve SDP.

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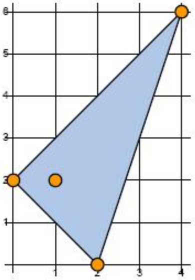

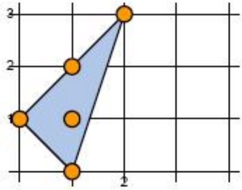

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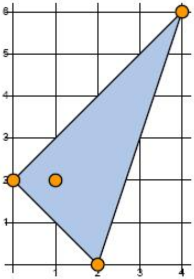

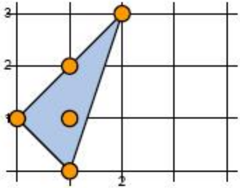

- We need all of $\frac{1}{2}N(p)$.

- **Example:** $p(x_1, x_2) = 1 + x_1x_2 + x_2x_1 + x_2x_1^2x_2$ has SOHS decomp. (p is SOHS) since

$$p(x_1, x_2) = \begin{bmatrix} 1 \\ x_1x_2 \end{bmatrix}^* \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x_1x_2 \end{bmatrix}$$

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- Newton polytope idea does not work (NC polynomial can be represented only by vectors from $\mathbb{Z}_n^d$, but multiplication of monomials is not addition of vectors).

1. **Example:** Find vector $M(p)$ and PSD matrix Q such that

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2. Take all possible monomials of degree $\leq d$:

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3. Do the **reduction**:
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   - x_2x_1 can not appear;
4. Construct and solve the SDP: if $\bar{M}(p) = \left[\ 1,\ x_1,\ x_2,\ x_1x_2\ \right]$, then SDP contains equations:

$$\begin{array}{rrcl}
1: & q_{1,1} & = & 1 \\
x_1x_2: & q_{1,4} + q_{2,3} & = & 1 \\
x_2x_1: & q_{3,2} + q_{4,1} & = & 1 \\
x_2x_1^2x_2: & q_{4,4} & = & 1 \\
x_1^2: & q_{2,2} & = & 0 \\
x_2^2: & q_{3,3} & = & 0
\end{array}$$

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- Let $p(x,y) = x^2 - x^{10}y^{20}x^{11} - x^{11}y^{20}x^{10} + x^{10}y^{20}x^{20}y^{20}x^{10}$.
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- It is easy to see that

$$p = (x - x^{10}y^{20}x^{10})^*(x - x^{10}y^{20}x^{10}) \in \text{SOHS}.$$

- **right chip function:** $r_1(x_1x_1x_2x_2x_2\mathbf{x_1}) = \mathbf{x_1}$, $r_2(x_1x_1x_2x_2\mathbf{x_2x_1}) = \mathbf{x_2x_1}$.

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- **Newton chip method** (NCM)

INPUT: $p \in \mathbb{R}\langle \bar{x}\rangle$ *with* $\deg p = 2d$, $p = \sum_{w\in\langle \bar{x}\rangle} a_w w$,, *where* $a_w \in \mathbb{R}$.

Define the **support** *of* p: $\mathcal{W}_p := \{w \in \langle \bar{x}\rangle \mid a_w \neq 0\}$.

Step 1: $M(p) := \varnothing$.

Step 2: For every $w = w^* \in \mathcal{W}_p$:

Substep 2.1 For $0 \leq i \leq \frac{\deg w}{2}$: $M(p) := M(p) \cup \{r_i(w)\}$.

OUTPUT: $M(p)$.

Newton chip method to get $M(p)$

- **right chip function:** $r_1(x_1x_1x_2x_2x_2\mathbf{x_1}) = \mathbf{x_1}$, $r_2(x_1x_1x_2x_2\mathbf{x_2x_1}) = \mathbf{x_2x_1}$.
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- If $p(x,y) = x^2 - x^{10}y^{20}x^{11} - x^{11}y^{20}x^{10} + x^{10}y^{20}x^{20}y^{20}x^{10}$, then NCM gives $M(p)$ of length 41

Augmented Newton Chip method

1. Preprocessing: given $p \in \mathbb{R}\langle \bar{x} \rangle$ with degree $2d$
   - If p is not symmetric, then p is not SOHS
   - If no monomial of minimum or maximum degree is symmetric, then p is not SOHS
2. Compute $M(d)$ using NCM.
3. Iteratively delete all monomials from $M(d)$ which yield constraints of form $q_{i,i} = 0$.

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3. Iteratively delete all monomials from $M(d)$ which yield constraints of form $q_{i,i} = 0$.
4. Aug. NCM gives for $p(x,y) = x^2 - x^{10}y^{20}x^{11} - x^{11}y^{20}x^{10} + x^{10}y^{20}x^{20}y^{20}x^{10}$ the following: $M(p) = [x \;\; x^{10}y^{20}x^{10}]$ of length 2

- Authors: I. Klep in J. Povh.
- 1st version close to publication
- We also implemented **basic operations**, **optimization** and **convexity** above non-commutative polynomials.
- We plug in **SDPT3** or **sedumi** solver (new SDP solver seems to be unnecessary.)
- Demonstration

$$\begin{aligned} p(x_1, x_2) &= 1 + 3x_1^2 - x_1x_2 - 6x_1x_2x_1 + 3x_1x_2^2x_1 - x_2x_1 + x_2x_1^2x_2 \\ &= (1 - x_1x_2)^*(1 - x_1x_2) + 3(x_1 - x_2x_1)^*(x_1 - x_2x_1) \end{aligned}$$

- Forthcoming paper: J. Povh, I. Klep: Non-commutative sums of squares (2008).