Approximating Non-convex Quadratic Programs by Semidefinite and Copositive Programming[1]

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Abstract: Finding an optimum of a non-convex quadratic function is in general a very difficult task even if the feasible set is a polyhedron.

We show that when the feasible set of a quadratic problem consists of matrices from $\mathbb{R}^{n\times k}$ which have orthogonal columns, then we can transform the quadratic problem into a semidefinite program in matrices of order kn, which has the same optimal value.

This opens new possibilities to get good lower bounds for several problems from combinatorial optimization, like the *quadratic assignment problem* (QAP) and the *graph partitioning problem* (GP). In particular we show how to improve significantly the well-known Hoffman-Wielandt eigenvalue lower bound for QAP and the Donath-Hoffman eigenvalue lower bound for GP by semidefinite programming.

Finally we show how to rewrite some problems from combinatorial optimization as linear programs over the cone of completely positive matrices.

Keywords: semidefinite programming, copositive programming, eigenvalue bound, quadratic assignment problem, graph partitioning problem.

1 Introduction

Non-convex quadratic problems are a common research topic since they appear very often in combinatorial optimization. They are in general very hard, since already the following simple non-convex quadratic problem

$$\min\{x^T Qx : x \in \mathbb{R}^n_+, \sum_i x_i = 1\} \tag{1}$$

is NP-hard to solve. More specifically, when $Q = A + I$ and A is the adjacency matrix of a graph G, then the optimal value of (1) yields the stability number of G (see [8]), which is NP-hard to compute. In this paper we consider more general non-convex quadratic programs:

$$(QP) \quad \min\{\text{trace}(X^T AXB) : X \in \mathbb{R}^{n\times k}_+, X^T X = M, Q(X) = q\},$$

where A and B are arbitrary symmetric matrices, M is a diagonal matrix and $Q(X) = q$ denotes some additional quadratic constraints. Several well-known NP-hard problems can be restated in the form (QP), e.g. the quadratic assignment problem, the graph partitioning problem and with some small modifications also the problem (1) (see

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[12]).

The problems listed above are very tough and there is no polynomial time algorithm which finds the optimal solution of these problems (unless P=NP). Optimal solutions are often computed with a branch and bound algorithm, which has an exponential time complexity. The efficiency of this algorithm strongly depends on the quality of upper and lower bounds for the optimal value of the problem. Upper bounds we get with any heuristic, while computing lower bounds typically consists in relaxing some hard constraint and computing the optimal value of the relaxed problem.

Many researchers studied relaxations which yield spectral lower bounds. Hoffman and Wielandt [6] established an eigenvalue lower bound for the optimal value of (QP) for the case when the feasible set consists of non-negative square orthonormal matrices. They dropped the sign constraint and computed the optimal value of the relaxed problem, which is determined by the eigenvalues of A and B, see Section 2. This is also known as the eigenvalue lower bound for the quadratic assignment problem.

Donath and Hoffman [4] presented an eigenvalue lower bound for the graph partitioning problem, which is another special case of (QP). They again relaxed the original problem, which is NP-hard, by ignoring the sign constraint and reformulated the obtained problem as an eigenvalue optimization problem.

Helmberg et al. [5] and Rendl and Wolkowicz [14] studied the projected eigenvalue lower bounds for the minimal cut problem in graph and graph partitioning problems.

Anstreicher and Wolkowicz [1] reformulated the Hoffman-Wielandt and the Donath-Hoffman lower bounds as optimal values of semidefinite programs. A similar result was obtained by Povh and Rendl for the eigenvalue lower bound from [5], see [10]. These results are very important because further strengthenings of the eigenvalue lower bounds lead to untractable problems, while the semidefinite reformulation enables adding additional constraints and therefore improving the lower bounds. The semidefinite formulations for the Hoffman-Wielandt and the Donath-Hoffman lower bounds are semidefinite programs in matrix variables of dimension $n^2 \times n^2$.

The main contribution of the present paper is a representation theorem in Section 2, which generalizes the Anstreicher-Wolkowicz result from [1] and yields much smaller semidefinite programs for the Donath-Hoffman lower bound. In Section 3 we present the implications of the theorem from Section 2 on computing lower bounds for the quadratic assignment problem and the graph partitioning problem. We propose some new constraints which significantly improve the quality of the lower bounds and demonstrate this effect on instances of medium size. Finally we present in Section 4 how to reformulate a family of non-convex quadratic problems, including the quadratic assignment problem and the graph partitioning problem, as a linear program over the cone of completely positive matrices. The proof of this result and applications of the result are omitted and will be reported elsewhere (see also [12]).

1.1 Notation

We denote the ith standard unit vector by e_i. The vector of all ones is $u_n \in \mathbb{R}^n$ (or u if the dimension n is obvious). The square matrix of all ones is J_n (or J), the identity matrix is I and $E_{ij} = e_i e_j^T$.

In this paper we consider the following sets of matrices:

- The vector space of real symmetric $n \times n$ matrices: $\mathcal{S}_n = \{X \in \mathbb{R}^{n \times n} : X = X^T\}$

• the cone of $n \times n$ positive semidefinite matrices: $\mathcal{S}_n^+ = \{X \in \mathcal{S}_n : y^T X y \geq 0, \forall y \in \mathbb{R}^n\}$,

• the cone of $n \times n$ copositive matrices: $\mathcal{C}_n = \{X \in \mathcal{S}_n : y^T X y \geq 0, \forall y \in \mathbb{R}_+^n\}$,

• the cone of $n \times n$ completely positive matrices:

$$\mathcal{C}_n^* = \{X = \sum_{i=1}^{k} y_i y_i^T, k \geq 1, y_i \in \mathbb{R}_+^n, \forall i = 1, \ldots, k\}.$$

We also use $X \succeq 0$ for $X \in \mathcal{S}_n^+$. A linear program over $\mathbb{R}_+^n$ is called a linear program, a linear program over $\mathcal{S}_n^+$ is called a semidefinite program while a linear program over $\mathcal{C}_n$ or $\mathcal{C}_n^*$ is called a copositive program.

The sign $\otimes$ stands for the Kronecker product. When we consider the matrix $X \in \mathbb{R}^{m \times n}$ as a vector from $\mathbb{R}^{mn}$, we write this vector as $\operatorname{vec}(X)$ or x. For $u, v \in \mathbb{R}^n$ we define $\langle u, v \rangle = u^T v$ and for $X, Y \in \mathbb{R}^{m \times n}$ we set $\langle X, Y \rangle = \operatorname{trace}(X^T Y)$. For matrix columns and rows we use the matlab notation. Hence $X(i,:)$ and $X(:,i)$ stand for ith row and column, respectively. If $a \in \mathbb{R}^n$, then $Diag(a)$ is an $n \times n$ diagonal matrix with a on the main diagonal and $\operatorname{diag}(X)$ is the main diagonal of a square matrix X.

For a matrix $Z \in \mathcal{S}_{kn}$ we often use the following block notation:

$$Z = \begin{bmatrix} Z^{11} & \cdots & Z^{1k} \\ \vdots & \ddots & \vdots \\ Z^{k1} & \cdots & Z^{kk} \end{bmatrix}, \tag{2}$$

where $Z^{ij} \in \mathbb{R}^{n \times n}$.

When P is the name of the optimization problem, then OPT_P denotes its optimal value.

2 Semidefinite programming relaxations for QP

Hoffman and Wielandt [6] showed that

$$OPT_{HW} = \min\{\langle X, AXB \rangle : X \in \mathbb{R}^{n \times n}, X^T X = I\} = \langle \lambda, \sigma \rangle_- \tag{3}$$

where λ and σ are the vectors of eigenvalues of A and B, respectively, and $\langle \lambda, \sigma \rangle_-$ denotes the scalar product, where we first sort the components of λ increasingly and the components of σ decreasingly. OPT_{HW} is a lower bound for OPT_{QP}, since the problem (3) is obtained from QP by omitting sign constraints and the quadratic constraint $Q(X) = q$.

Anstreicher and Wolkowicz [1] formulated this lower bound as the optimal value of a semidefinite program. They added in problem (3) the redundant constraint

$XX^T = I$ and then considered the Lagrangian dual of the problem, which is the semidefinite program from below. They showed that strong duality holds for this case, hence we have

$$OPT_{HW} \;=\; \max\{\text{trace}(S)+\text{trace}(T): S\in\mathcal{S}, T\in\mathcal{S}, S\otimes I + I\otimes T \preceq B\otimes A\}. \qquad (4)$$

In this section we extend this result to a more general case when the matrices $X\in\mathbb{R}^{n\times k}$ still have orthonormal columns but are not square. In this case we have $X^T X = I_k$ but the other constraint $XX^T = I_n$ is not satisfied, if $k<n$. Therefore we can not repeat the Anstreicher-Wolkowicz procedure. The following lemma shows how the constraint $XX^T = I_n$ should be generalized to close the duality gap.

Lemma 1 *If* $X\in\mathbb{R}^{n\times k}$ *and* $X^T X = I_k$*, then* $XX^T \preceq I_n$.

Proof: Let us fix X. We can complete the columns of X into an orthonormal basis $(u_1,u_2,\ldots,u_n)$ of the space $\mathbb{R}^n$, hence for $1\le i\le k$ we have $u_i = X(:,i)$. Let $u=\sum_{i=1}^n \alpha_i u_i$. We have $u^T(I-XX^T)u=\sum_{i=k+1}^n \alpha_i^2 \ge 0$, hence $I-XX^T\succeq 0$.

Now we can prove the theorem.

Theorem 2 *If* $A\in\mathcal{S}_n$ *and* $B\in\mathcal{S}_k$ *then*

$$\begin{aligned}&\min\{\langle X, AXB\rangle : X\in\mathbb{R}^{n\times k}, X^T X = I_k\} =\\ &\max\{\text{trace}(S)-\text{trace}(T): S\in\mathcal{S}_k, T\in\mathcal{S}_n^+, S\otimes I_n - I_k\otimes T\preceq B\otimes A\}\end{aligned} \qquad (5)$$

Proof: The proof mostly consists of extensions of the ideas from [1]. Let OPT_L and OPT_R be the optimal values of the left and the right problem in (5), resp. First we show that the second problem is the Lagrangian relaxation of the first problem, if we add the seemingly redundant constraint $XX^T\preceq I$, then we reduce it to a linear program and finally prove that the optimal value of the dual linear program is

$$\sum_i \sigma_i\lambda_{\varphi(i)}$$

where φ is some injection from $\{1,\ldots,k\}$ into $\{1,\ldots,n\}$ and λ,σ are vectors with the eigenvalues of A and B, resp. This is at least OPT_L [5, Theorem 5] and completes the chain of inequalities.

We introduce the dual variable S for the constraint $X^T X = I_k$ and the dual variable T for the newly added constraint $XX^T\preceq I_n$. Clearly $S\in\mathcal{S}_k$, $T\in\mathcal{S}_n^+$ and we have

$$\begin{aligned} OPT_L &= \min\{\langle X, AXB\rangle : X \in \mathbb{R}^{n\times k}, X^T X = I_k, XX^T \preceq I_n\} \\ &= \min_{X\in\mathbb{R}^{n\times k}} \{ \max_{S\in\mathcal{S}_k, T\in\mathcal{S}_n^+} \{\langle X, AXB\rangle + \langle S, I_k - X^T X\rangle - \langle T, I_n - XX^T\rangle\}\} \\ &\geq \max_{S\in\mathcal{S}_k, T\in\mathcal{S}_n^+} \{\mathrm{trace}(S) - \mathrm{trace}(T) + \min_{x\in\mathbb{R}^{nk}} x^T(B\otimes A - S\otimes I_n + I_k \otimes T)x\} \\ &= \max\{\mathrm{trace}(S) - \mathrm{trace}(T) : S\in\mathcal{S}_k, T\in\mathcal{S}_n^+, S\otimes I_n - I_k\otimes T \preceq B\otimes A\} \\ &= OPT_R. \end{aligned}$$

The first inequality follows from exchanging min and max and the last equality is due to the inner minimization problem, which is a quadratic unconstrained problem and is therefore bounded from below if and only if its Hessian $B\otimes A - S\otimes I_n + I_k\otimes T$ is positive semidefinite. We used the fact that

$$\langle X, AXB\rangle = x^T(B\otimes A)x, \quad for\ x = \mathrm{vec}(X).$$

We show that there is an equality above by transforming the last semidefinite program into a linear program. Since A and B are symmetric, we can find an orthonormal decomposition $A = P\Lambda P^T$ and $B = Q\Sigma Q^T$, where $\Lambda = \mathrm{Diag}(\lambda)$, $\Sigma = \mathrm{Diag}(\sigma)$, vectors λ, σ are as above and P, Q are matrices, whose columns are eigenvectors of A and B, respectively. We can write

$$\begin{aligned} OPT_R &= \max\{\mathrm{trace}(S) - \mathrm{trace}(T) : S\in\mathcal{S}_k, T\in\mathcal{S}_n^+, \Sigma\otimes\Lambda - S\otimes I_n + I_k\otimes T \succeq 0\} \\ &= \max\{u_k^T s - u_n^T t : s\in\mathbb{R}^k, t\in\mathbb{R}_+^n, s_i - t_j \leq \sigma_i\lambda_j, \forall i,j\} \\ &= \min\{\sum_{i,j}\sigma_i\lambda_j z_{ij} : y\in\mathbb{R}_+^n, Z\in\mathbb{R}_+^{k\times n}, Zu_n = u_k, Z^T u_k + y = u_n\}. \end{aligned}$$

The first equality in the expression above follows from the fact that the cost function depends only on diagonal entries of the matrices S and T, so we may ignore all non-diagonal entries and write $s = \mathrm{diag}(S)$ and $t = \mathrm{diag}(T)$. The last optimization problem is a dual linear program to the last but one problem. We should note that the system matrix in the last linear program is totally unimodular, hence there exists (see [9]) an integer optimal solution

$$(Z^*, y^*) \in \mathbb{R}_+^{k\times n} \times \mathbb{R}_+^n.$$

The matrix Z^* is therefore a 0-1 matrix and defines an injection $\varphi^* : \{1,\ldots,k\} \to \{1,\ldots,n\}$ with $\varphi^*(i) = j \Leftarrow z_{ij}^* = 1$. This means that we have proved

$$OPT_R = \sum_{i=1}^k \sigma_i\lambda_{\varphi^*(i)} \geq \min\{\sum_i \sigma_i\lambda_{\varphi(i)} : \varphi\ injection : \{1,\ldots,k\}\to\{1,\ldots,n\}\}.$$

The optimal value of the right-hand side problem above is exactly OPT_L and from all relations from the beginning we can conclude that $OPT_L = OPT_R$. □

In the following lemma we analyze the SDP representations (4) and (5) for the case $k = n$.

Lemma 3 *If $k = n$, then (4) and (5) are equivalent.*

Proof: Since any feasible solution for (5) is feasible for (4) and gives the same objective value, we need to prove only that if $(S^*, T^*) \in \mathcal{S}_k \times \mathcal{S}_n$ is optimal for (4), then it gives a feasible solution for (5) with the same objective value. If $T^* \succeq 0$, then we are done, otherwise we define $\hat{S} = S^* - \lambda_{\min}(T^*)I$ and $\hat{T} = T^* - \lambda_{\min}(T^*)I$. Obviously is $\hat{T} \succeq 0$, $(\hat{S}, \hat{T})$ is feasible for (5) and $\text{trace}(\hat{S}) - \text{trace}(\hat{T}) = \text{trace}(S^*) - \text{trace}(T^*)$. □

3 New SDP lower bounds for the quadratic assignment and the graph partitioning problem

In this section we use the result from Theorem 2 to find better lower bounds for the quadratic assignment and the graph partitioning problem, based on semidefinite programming. Firstly we write a dual semidefinite program for the semidefinite program (5). We introduce dual variables $X \in \mathcal{S}_n^+$ and $Y \in \mathcal{S}_{kn}^+$ for both semidefinite constraints to obtain the following dual semidefinite program

$$
(QP_{SDP}) \qquad \begin{array}{lrclrcl} \min & \langle B \otimes A, Y \rangle & & & & & \\ \text{s.t.} & Y & \in & \mathcal{S}_{kn}^+, & X & \in & \mathcal{S}_n^+ \\ & \sum_i Y^{ii} + X & = & I, & \langle I, Y^{ij} \rangle & = & \delta_{ij}, \quad 1 \le i, j \le k. \end{array}
$$

Here is δ_{ij} the Kronecker's δ. We denote the optimal value of this semidefinite program by OPT_{SDP}. Note that the semidefinite program (5) has strictly feasible points (e.g. $T = I_n$ and $S = \lambda_{\min}(B \otimes A) \cdot I_k$ are strictly feasible), hence it's optimal value is equal to the OPT_{SDP}.

Remark 1 *If $k = n$, then from the last two constraints it follows* $\text{trace}(X) = \text{trace}(I) - \text{trace}(\sum_i Y^{ii}) = 0$. *Since $X \succeq 0$ we have $X = 0$, hence we can eliminate X from QP_{SDP}.*

In the rest of the section we illustrate, how the lower bound OPT_{SDP} improves towards the optimum if we add to the QP_{SDP} new constraints.

3.1 Quadratic assignment problem

The Quadratic Assignment Problem (QAP) can be stated in the following way. Let Π be the set of $n \times n$ permutation matrices (a matrix X is a permutation matrix if it corresponds to some permutation ϕ, i.e. $x_{ij} \in \{0,1\}$ and $x_{ij} = 1 \Leftarrow \phi(i) = j$). For given real symmetric $n \times n$ matrices A and B we want to find a permutation matrix $X \in \Pi$ which gives

$$
(QAP) \qquad OPT_{QAP} = \min\{\langle X, AXB \rangle : X \in \Pi\}.
$$

The QAP is nowadays widely considered as a classical combinatorial optimization problem, but it is also known as a generic model for various real-life problems, see the QAP library [3] for more references on QAP. The QAP is well known to be NP-hard, and even approximating the OPT_{QAP} within a constant factor is an NP-hard problem. The computational effort to solve the QAP is very likely to grow exponentially with the problem size, and problems of size $n \geq 25$ are currently considered as large instances.

The most recent and promising trends of research to find good lower bounds for OPT_{QAP} are based on semidefinite programming. Zhao et al., Sotirov and Rendl and Povh and Rendl[15, 13, 11] lifted the problem from the vector space $\mathbb{R}^{n\times n}$ to $\mathcal{S}^{+}_{n^2+1}$ or $\mathcal{S}^{+}_{n^2}$ and formulated several SDPs which give increasingly tight lower bounds for the QAP. They used several methods to solve these SDP. The computational results show that these lower bounds are among the strongest but also the most expensive to compute (in practice the could solve these SDPs for $n \leq 35$).

In this subsection we present a new lower bound for the QAP, based on semidefinite programming, which is obtained by strengthening the Hoffman-Wielandt lower bound. Indeed, problem (3) may be considered as a relaxation for the QAP, since the set of permutation matrices can be defined also as $\Pi = \{X \in \mathbb{R}^{n\times n}_{+} : X^T X = I\}$.

Considering Lemma 3 and Remark 1, the following SDP yields a lower bound for OPT_{QAP}, which is at least as strong as the Hoffman-Wielandt lower bound:

$$(QAP_{SDP}) \qquad \begin{array}{llllllll} \min & \langle B \otimes A, Y\rangle & & & & & & \\ \text{s.t.} & Y & \in & \mathcal{S}^{+}_{n^2}, & \langle I, Y^{ij}\rangle & = & \delta_{ij}, & 1 \leq i, j \leq n \\ & \sum_i Y^{ii} & = & I, & \mathcal{A}(Y) & = & a. & \end{array}$$

The constraint $\mathcal{A}(Y) = a$ must belong to a constraint, which is valid for permutation matrices and must not break the strong duality property for this semidefinite program. We add to QAP_{SDP} the following constraint:

$$\langle J_{n^2}, Y\rangle = n^2. \tag{6}$$

It represents in the original problem the constraint that the sum of all components of an $n \times n$ matrix is n, which is satisfied by all permutation matrices.

Table 1 demonstrates the improvement, obtained by adding (6) to the Hoffman-Wielandt eigenvalue lower bound. The first column contains the name of the problem instance, which also tells us the size of the instance. The second column contains the Hoffman-Wielandt eigenvalue lower bound and in the third column we see the new lower bound, denoted by OPT_{new}, which is obtained by adding constraint (6) to QAP_{SDP}. The column OPT_{best} contains the best known SDP lower bound for the problem instance (this is the lower bound, based on the QAP_{R_3} model from [13]) and in the last column we have the optimal value of the QAP instance. Data for the last two columns are taken from [13].

name	OPT_{HW}	OPT_{new}	OPT_{best}	OPT_{QAP}
nug15	-1746	981	1122	1150
nug20	-3198	2245	2451	2570
nug25	-4725	3235	3535	3744
nug30	-10965	5305	5803	6124

Table 1: The new lower bound lb_{new} improves the eigenvalue lower bound OPT_{HW} significantly

We can see that the single constraint (6) improves the eigenvalue lower bound significantly. We also point out that the resulting SDP is still quite simple (it has $\mathcal{O}(n^2)$ linear constraints) comparing to the SDP underlaying the best lower bound, which contains $\mathcal{O}(n^4)$ linear constraints.

3.2 The Graph partitioning problem

The Graph partitioning problem (GP) is a classical problem from combinatorial optimization. Given a graph $G=(V,E)$ with $|V|=n$, a number of partitions $k>1$ and a vector $m=(m_1,m_2,\ldots,m_k)\in\mathbb{N}^k$ with $1\le m_1\le m_2\le\ldots\le m_k$, $\sum_i m_i=n$, we are interested in a partition $(S_1,S_2,\ldots,S_k)$ of the vertex set V such that $|S_i|=m_i$ and the total number of cut edges (i. e., edges between different sets) is minimum.

We may represent any partition into k blocks with prescribed sizes by a matrix $X\in\{0,1\}^{n\times k}$, where $x_{ij}=1$ if and only if the ith vertex belongs to jth set. With this notation the total number of cut edges is exactly $0.5\langle X,AXB\rangle$, where A is the adjacency matrix of the graph (i.e. $a_{ij}=1$ if (ij) is an edge and $a_{ij}=0$ otherwise) and $B=J_k-I_k$. If L is Laplacian matrix of a graph, then it holds $0.5\langle X,AXB\rangle=0.5\langle X,LX\rangle$.

The set of all partition matrices may be described as

$$\{X\in\mathbb{R}_+^{n\times k}: X^TX=M, \operatorname{Diag}(XX^T)=u_n\}, \tag{7}$$

where $M=\operatorname{Diag}(m)$. We may describe the partition matrices also by other equations, but the equations from (7) are the most convenient for our bounding procedure.

Graph partitioning problem may be formulated as

$$(GP) \qquad OPT_{GP}=\min\{\langle X,LX\rangle: X\in\mathbb{R}_+^{n\times k}, X^TX=M, \operatorname{Diag}(XX^T)=u_n\}.$$

If we ignore the sign constraint and the last constraint and use the substitution $Y=XM^{-1/2}$, then we get the following lower bound for OPT_{GP}:

$$\min\{\langle Y,LYM\rangle: Y\in\mathbb{R}^{n\times k}, Y^TY=I_k\}. \tag{8}$$

We may apply Theorem 2 and the dualization procedure from the beginning of the section. The semidefinite program QP_{SDP}, where we replace $B \otimes A$ by $M \otimes L$, yields the eigenvalue lower bound for OPT_{GP}, denoted by OPT_{eig}, which seems to be much better than the Hoffman-Wielandt lower bound for QAP, as we can see, if we compare results from Tables 1 and 2 (the Hoffman-Wielandt lower bound for QAP is on these test instances always negative and therefore of no use).

name	OPT_{eig}	OPT_{DH}	OPT_{new}
g50.02	51	72	83
g50.04	177	213	247
g50.06	307	350	406
g50.08	502	536	623

Table 2: Lower bounds for graph partitioning problem, where $m = (5,10,15,20)$

We can strengthen the lower bound OPT_{eig} by adding further constraints to (8). If we add constraint $u_n = \text{Diag}(XX^T) = \text{Diag}(YMY^T)$ and repeat the dualization procedure, we obtain in the semidefinite program QP_{SDP} the constraint

$$\sum_i m_i \cdot \text{diag}(Y^{ii}) = u_n. \tag{9}$$

The optimal value of the resulting semidefinite program is exactly the Donath-Hoffman lower bound for OPT_{GP}, denoted by OPT_{DH} (see [1, 12]). We also demonstrate a further improvement of this lower bound by adding the constraint $\langle J_{kn}, Y\rangle = n^2$ to the semidefinite program, underlaying the OPT_{DH} (this is constraint (6), adapted to the graph partitioning problem). The new lower bound is denoted by OPT_{new}. Table 2 demonstrates the quality of these lower bounds on 4 random graph instances with 50 nodes, where the probability that there exists an edge between two nodes varies from 0.2 to 0.8 (the probability is included in the name of the instance).

We can see that these lower bounds are closer comparing to lower bounds for QAP. However, these results again demonstrates that adding few constraints to a semidefinite formulation of an eigenvalue lower bound does not make the semidefinite problem much harder, but the optimal value is improved significantly.

4 Copositive programming approach to QP

When the feasible matrices for QP share the property that the sum of each row and column is fixed and prescribed, then we can rewrite QP as a linear program over the cone of completely positive matrices. More formally, let us consider the following set

$$\{X \in \mathbb{R}_+^{n\times k} : X^T X = M, Xu_k = a, X^T u_n = b\}, \tag{10}$$

where $a \in \mathbb{R}_+^n$ and $b \in \mathbb{R}_+^k$. We can square each of these equations and also multiply each row equation with each column equation:

$$(e_i X u_k)^2 = a_i^2,\ (e_j X^T u_n)^2 = b_j^2,\ (e_i X u_k)(e_j X^T u_n) = a_i b_j,\ 1 \le i \le n, 1 \le j \le k. \tag{11}$$

This gives the following description of set (10):

$$\{X \in \mathbb{R}_+^{n\times k} : X^T X = M, X \ \mathit{feasible\ for}\ ()\}. \tag{12}$$

If we repeat the procedure from Sections 2 and 3, we get finally the following linear program over the cone of completely positive matrices:

$$(QP_{CP}) \quad \begin{array}{lrcl} \min & \langle B\otimes A, Y\rangle & & \\ \text{s.t.} & Y & \in & \mathcal{C}_{kn}^* \\ & \langle I, Y^{ij}\rangle & = & m_i\delta_{ij}, \quad \langle J_k \otimes E_{ii}, Y\rangle = a_i^2 \\ & \langle E_{ii}\otimes J_k, Y\rangle & = & b_i^2 \quad \langle (e_j u_k^T)\otimes(u_n e_i^T), Y\rangle = a_i b_j \end{array}$$

We can prove (see [12]) that the optimal value of this copositive program is equal to the optimal value of the original quadratic problem. Moreover, we can show that

$$\mathrm{CONV}\{xx^T : x = \mathrm{vec}(X), X \ \mathit{belongs\ to}\ ()\} = \{Y \in \mathcal{C}_{kn}^* : Y \ \mathit{feasible\ for}\ QP_{CP}\}.$$

This does not make the problems tractable, but opens new possibilities to approximate the optimal value of the original problem, since any relaxation of the hard constraint $Y \in \mathcal{C}_{kn}^*$ yields a lower or upper bound for the optimum. Detailed analysis of this new approach has been done for quadratic assignment problem (see [11]) and min cut problem [10] and it has been shown that we can get very strong SDP lower bounds by using proper relaxation of the copositive constraint. The contribution of this representation to other combinatorial problems is still under research and will be reported elsewhere, see also the forthcoming dissertation [12].

Recently Burer [2] presented a procedure how to rewrite any nonconvex quadratic program having a mix of binary and continuous variables over a bounded feasible set as a linear program over the cone of completely positive matrices. This result is in many ways more general than the result mentioned above, but it is not clear how to include quadratic constraints.

5 Conclusions

Computing tight lower bounds for hard problems from combinatorial optimization is very important, especially if we plan to solve the problem by Branch and Bound method. In the paper we present how to improve eigenvalue lower bounds for some problems, where the feasible set consists of orthogonal matrices, by using semidefinite programming. In particular we rewrite the quadratic problem over the set of orthogonal matrices as semidefinite program and then present which constraints we may add to the semidefinite program in order to get good lower bounds for quadratic assignment problem and graph partitioning problem. Preliminary computational results show the strong potential of this approach.

Finally we explain how to reformulate some non-convex quadratic programs as linear program over the cone of completely positive matrices. This result again confirms the importance of copositive programming in combinatorial optimization, which was revealed by de Klerk and Pasechnik [7]. Importance of this copositive representation has been analyzed for quadratic assignment problem [11] and min-cut problem [10], while its contribution to solving other problems from combinatorial optimization is still an open question (see also the dissertation [12]).

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