SOHS decomposition of a non-commutative polynomial: find it by NCSOStools

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Outline

1. Motivation.
2. SOHS decomposition of real non-commutative polynomials.
3. Newton chip method.
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- $\mathbb{R}\langle \bar{x} \rangle = \mathbb{R}\langle x_1, \dots, x_n \rangle$ ... algebra of real polynomials in n non-commutative variables $\bar{x} = (x_1, \dots, x_n)$ (i.e.: $x_i x_j \neq x_j x_i, \forall i, j.$)
- We equip $\mathbb{R}\langle \bar{x} \rangle$ with the **involution** $*$:

$$(x_1 x_2 - x_1^2 x_3)^* = x_2 x_1 - x_3 x_1^2.$$

- We assume $x_i^* = x_i$ (symmetric variables).
- $\mathbb{R}\langle \bar{x} \rangle$ is the $*$-algebra freely generated by n symmetric letters.
- $\operatorname{Sym} \mathbb{R}\langle \bar{x} \rangle$ - the set of all **symmetric elements**:

$$\operatorname{Sym} \mathbb{R}\langle \bar{x} \rangle = \{ f \in \mathbb{R}\langle \bar{x} \rangle \mid f = f^* \}.$$

- **hermitian square**: NC polynomial of the form $g^* g$.
- **sums of hermitian squares (SOHS)**: $\Sigma^2 \subsetneq \operatorname{Sym} \mathbb{R}\langle \bar{x} \rangle$.

- Given $f \in \mathbb{R}\langle \bar{x} \rangle$, we have
  - $\deg f$ (**degree** of f): length of the longest word in f
  - $\deg_i f$: **degree** of f in x_i.
  - $\operatorname{mindeg} f$: length of the shortest word appearing in f
  - Likewise: $\operatorname{mindeg}_i f$
  - If x_i does not occur in f, then $\operatorname{mindeg}_i f = 0$.
- **Example**: if $f = x_1^3 + 2x_1x_2x_3 - x_1^2x_4^2$, then

$$\deg f = 4, \quad \deg_1 f = 3, \quad \deg_2 f = \deg_3 f = 1, \quad \deg_4 f = 2,$$

$$\operatorname{mindeg} f = 3, \quad \operatorname{mindeg}_1 f = 1,$$

$$\operatorname{mindeg}_2 f = \operatorname{mindeg}_3 f = \operatorname{mindeg}_4 f = 0.$$

Motivation for SOHS decomposition

- **No MATLAB** package for handling the non-commutative polynomials.
- Let $f \in \mathbb{R}\langle \bar{x} \rangle$. **If there exist** polynomials $q_1, \ldots, q_k \in \mathbb{R}\langle \bar{x} \rangle$ such that $f = \sum_i q_i^* q_i$, **then** $f \succeq 0$.
- Polynomials q_i are **SOHS** decomposition for f.
- We can say more:
  Theorem (J. W. Helton, Annals of Mathematics, 2002):
  $\boxed{f \in \mathbb{R}\langle \bar{x} \rangle \text{ is SOHS} \Leftrightarrow f \succeq 0}$
  whenever we replace x_i by symmetric matrices X_i of dimension $k \times k$, for any $k \geq 1$.

More motivation for SOHS decomposition

- **Bessis - Moussa - Villani (BMV) conjecture (1975):**
  For symmetric matrices A, B with B positive semidefinite, the function
  $$\Phi^{A,B} : \mathbb{R} \to \mathbb{R}, t \mapsto \mathrm{trace}(e^{A-tB})$$
  is the Laplace transform of a positive measure $\mu^{A,B}$ on $[0, \infty)$:
  $$\mathrm{trace}(e^{A-tB}) = \int_0^\infty e^{-tx} d\mu^{A,B}(x).$$
- **Equivalently** (Lieb–Seiringer, 2004): The polynomial $\mathrm{trace}((A + tB)^m) \in \mathbb{R}[t]$ has only nonnegative coefficients whenever A, B are PSD of order s, for all m.
- Conjecture proved for $m \leq 13$ by Klep and Schweighofer (2008) using SOHS concept.
- **Idea**: coefficients in $(A + tB)^m$ are (cyclically equivalent to) SOHS.

Proposition

Suppose $f \in \operatorname{Sym}\mathbb{R}\langle \bar{x}\rangle$ is of degree $\leq 2d$. Then $f \in \Sigma^2$ if and only if there exists a positive semidefinite (PSD) matrix G satisfying

$$f = W_d^* G W_d, \tag{1}$$

where W_d is a vector consisting of all words in $\langle \bar{x}\rangle$ of degree $\leq d$.

Remark: Given such a PSD matrix G with rank r, the SOHS decomposition is

$$f = \sum_{i=1}^{r} g_i^* g_i, \tag{2}$$

where $g_i = H(i,:)W_d$, $G = H^T H$.

Proposition

Suppose $h \in \mathrm{Sym}\,\mathbb{R}\langle\bar{x}\rangle$ is **homogeneous** of degree $2d$ and let V_d be a vector consisting of all words in $\langle\bar{x}\rangle$ of degree d.

(a) h has essentially a unique Gram matrix, i.e., there is a unique self-adjoint matrix G satisfying

$$h = V_d^* G V_d. \tag{3}$$

(b) $h \in \Sigma^2$ if and only if G in (3) is positive semidefinite.

Remark: This is **not true** in the commutative case: $f = x^4 + y^4 + 2x^2y^2$.

- **Corollary**: $f = \sum_{w \in \langle \bar{x} \rangle} a_w w \in \mathrm{Sym}\,\mathbb{R}\langle \bar{x} \rangle$ is SOHS **iff** exists $G \succeq 0$ such that:

$$\sum_{\substack{p,q \in W_d \\ p^* q = w}} G_{p,q} = a_w, \quad \forall w \in \langle \bar{x} \rangle, a_w \neq 0.$$

- "Is f SOHS?" is SDP feasibility problem.
- Resulting SDP

$$\begin{array}{llcll} \inf & \langle I, G \rangle & & & \\ s.\,t. & \langle A_w, G \rangle & = & a_w + a_{w^*} & \forall w \in \{p^* q \mid p, q \in W_d\} \\ & G & \succeq & 0. & \end{array}$$

where

$$(A_w)_{u,v} = \begin{cases} 2; & \text{if } u^* v \in \{w, w^*\},\ u = v, \\ 1; & \text{if } u^* v \in \{w, w^*\},\ u \neq v, \\ 0; & \text{otherwise.} \end{cases}$$

Input: $f \in \operatorname{Sym}\mathbb{R}\langle\bar{x}\rangle$ *with* $\deg f \leq 2d$, $f = \sum_{w\in\langle\bar{x}\rangle} a_w w$,, *where* $a_w \in \mathbb{R}$.

STEP 1: *Construct* W_d.

STEP 2: *Construct data* A_i, b, C *corresponding to the SDP.*

STEP 3: *Solve the SDP to obtain* G. *If the SDP is not feasible, then* $f \notin \Sigma^2$; *stop.*

STEP 4: *Compute the Cholesky decomposition* $G = R^*R$.

Output: *Sum of hermitian squares decomposition of* f: $f = \sum_i g_i^* g_i$, *where* g_i *denotes the* i*-th component of* RW_d.

Algorithm 1: The Gram matrix method for finding SOHS decompositions

Example: Let

$$f = 1 + x_1x_2 + x_2x_1 + x_2x_1^2x_2$$

STEP 1: Take all possible monomials of degree ≤ 2:

$$W_2 = \left[\ 1,\ x_1,\ x_2,\ x_1^2, x_2^2,\ x_1x_2,\ x_2x_1\ \right]^*$$

STEP 2: Construct SDP with constraints

$$\begin{array}{rrcl}
1: & q_{1,1} & = & 1 \\
x_1x_2: & q_{1,7} + q_{6,1} + q_{2,3} & = & 1 \\
x_2x_1: & q_{1,6} + q_{3,2} + q_{7,1} & = & 1 \\
x_2x_1^2x_2: & q_{7,7} & = & 1 \\
x_1^2: & q_{2,2} + q_{1,4} + q_{4,1} & = & 0 \\
x_2^2: & q_{3,3} + q_{1,5} + q_{5,1} & = & 0 \\
x_1^4 & q_{2,2} & = & 0 \\
x_2^4 & q_{3,3} & = & 0 \\
x_1x_2^2x_1: & q_{6,6} & = & 0
\end{array}$$

STEP 3: **Optimal solution** of SDP:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T$$

Output: $f = (1 + x_1x_2)^*(1 + x_1x_2) \in \Sigma^2$.

- If $\deg f = 2d$, then W_d has length $N(n,d) := \sum_{k=0}^{d} n^k = \frac{n^{d+1}-1}{n-1}$ (**too large**)
- Let $f = x^2 - x^{10}y^{20}x^{11} - x^{11}y^{20}x^{10} + x^{10}y^{20}x^{20}y^{20}x^{10}$.
- The size of W_d is $2^{41} - 1$ (too large for today's SDP solvers)
- It is easy to see that

$$f = (x - x^{10}y^{20}x^{10})^*(x - x^{10}y^{20}x^{10}) \in \Sigma^2.$$

- Number of constraints:

$$\begin{aligned} m &= \operatorname{card}\{p^*q \mid p,q \in W_d,\ p^*q = q^*p\} + \\ &+ \frac{1}{2}\operatorname{card}\{p^*q \mid p,q \in W_d,\ p^*q \neq q^*p\} \\ &> \frac{1}{2}N(n,2d) = \frac{n^{2d+1}-1}{2(n-1)}. \end{aligned}$$

- **right chip function:** $r_1(x_1x_1x_2x_2x_2\mathbf{x_1}) = \mathbf{x_1}$, $r_2(x_1x_1x_2x_2\mathbf{x_2x_1}) = \mathbf{x_2x_1}$.
- **NEWTON CHIP METHOD** (NCM)

INPUT: $p \in \mathbb{R}\langle\bar{x}\rangle$ *with* $\deg p = 2d$, $p = \sum_{w\in\langle\bar{x}\rangle} a_w w$,, *where* $a_w \in \mathbb{R}$.
Define the **support** *of* p: $\mathcal{W}_p := \{w \in \langle\bar{x}\rangle \mid a_w \neq 0\}$.
Step 1: $W := \varnothing$.
Step 2: For every $w = w^* \in \mathcal{W}_p$:
Substep 2.1 For $0 \leq i \leq \frac{\deg w}{2}$: $W := W \cup \{r_i(w)\}$.
OUTPUT: W.

- If $f = x^2 - x^{10}y^{20}x^{11} - x^{11}y^{20}x^{10} + x^{10}y^{20}x^{20}y^{20}x^{10}$, then NCM gives W of length 41

Theorem

Suppose $f \in \mathrm{Sym}\,\mathbb{R}\langle \bar{x} \rangle$. *Then* $f \in \Sigma^2$ *if and only if there exists a positive semidefinite* G *satisfying*

$$f = W^* G W, \tag{4}$$

where W *is the output given by the Newton chip method.*

Proof ($\Leftarrow$) obvious.

($\Rightarrow$) Let $f = \sum_i q_i^* q_i$. Consider some $q_i = (\cdots + m + \cdots)$

- **If** $m^* m$ appears in f, we have $m \in W$.
- **Else** $m^* m = p^* q$ for $|p| > |q|$. We continue by considering $p^* p$.
- **Finally** we obtain r such that $r^* r$ **appears** in f and m is a **right chip** of r

Augmented Newton Chip method

1. **Preprocessing**: given $f \in \mathbb{R}\langle\bar{x}\rangle$ with degree $2d$
   - If f is not symmetric, then f is not SOHS
   - If no monomial of minimum or maximum degree is symmetric, then f is not SOHS
   - If no monomial of minimum or maximum degree in ith variable is symmetric, then f is not SOHS
2. **Compute** W using NCM.
3. **Keep** in W only monomials w such that

$$m_i \leq \deg_i w \leq M_i \forall i,\ m \leq \deg w \leq M,$$

where

$$m_i := \frac{\mathrm{mindeg}_i f}{2},\quad M_i := \frac{\deg_i f}{2} \quad m := \frac{\mathrm{mindeg} f}{2},\quad M := \frac{\deg f}{2}$$

4. Iteratively **delete** all monomials from W which yield constraints of form $g_{i,i} = 0$.

Aug. NCM gives for

$$f = x^2 - x^{10}y^{20}x^{11} - x^{11}y^{20}x^{10} + x^{10}y^{20}x^{20}y^{20}x^{10}$$

the following:

$$W = [x \;\; x^{10}y^{20}x^{10}]^*$$

.

- **Helton's theorem** (Helton, 2002) implies:

$$f(A_1, \dots, A_n) \succeq \varepsilon I$$

for $\varepsilon \in \mathbb{R}$ and for all self-adjoint matrices A_i of the same size **if and only if**

$$f - \varepsilon \in \Sigma^2.$$

- The largest such ε is obtained by solving the SDP

$$\begin{array}{ll} \sup & \varepsilon \\ \text{s.t.} & f - \varepsilon \in \Sigma^2. \end{array} \qquad (\mathrm{SDP_{min}})$$

- $(\mathrm{SDP_{min}})$ rewrites into

$$\begin{array}{lrcl} \sup & f_0 - \langle E_{11}, G\rangle & & \\ \text{s.t.} & f - f_0 & = & W^*(G - G_{11}E_{11})W \\ & G & \succeq & 0. \end{array}$$

- Authors: K. Cafuta, I. Klep, J. Povh.
- 1st version close to publication.
- We implemented **basic NC operations**, **searching for SOHS**, **optimization** and **convexity** above non-commutative polynomials.
- We use **SDPT3** or **sedumi** solver (potencial for new SDP solver like Boundary point method from Povh, Rendl and Wiegele, 2006.)

NCvars(varargin)

possible usage: NCvars(varargin), NCvars

example:

```
>> NCvars x y z

>> NCvars
   'x'    'y'    'z'
>> f=1-x*y;g=x*x+x*y+y*x+y*2;

>> -f'
ans = -1+y*x
>> g-f'*f
ans = -1+x*x+2*x*y+2*y*x-y*x*x*y+y*y
```

Power `f^4`.

Equal, not equal `f` == g or `f` = g.

NCsimplify

```
>> NCsimplify(x*x*x*x*y^2*x*x*y-x^2+x*x)
ans = x^4*y^2*x^2*y
```

NCexpand

```
>> NCexpand(x^4*y^2*x^2*y+x*x-x^2)
ans = x*x*x*x*y*y*x*x*y
```

call `[IsSohs,X,base,sohs,g]=NCsos(f,precision)`

- **description**: checks whether $f \in \Sigma^2$.
- **arguments:** f is an NCpoly. *Precision* - the smallest value that is considered to be nonzero.
- **output:**
  - $IsSohs == 1$ iff $f \in \Sigma^2$
  - X is a Gram matrix solution given by SDP solver.
  - *Base* is a list of monomials in the SOHS decomposition.
  - *Sohs* is a list of monomials m_i with $f = \sum_i m_i^* m_i$.
  - $g = \sum_i m_i^* m_i$.
- **possible usage:** $NCsos(f)$, $NCsos(f, precision)$

- **example:**

```
>> f=y*x^2*y-y*x*z+4*y*z^2*y-z*x*y+z^2;
>> [IsSohs,X,base,sohs,g]=NCsos(f)
*************** Polynomial is SOHS ************
IsSohs = 1
X = 1.0000   -0.0000   -1.0000
   -0.0000    4.0000    0.0000
   -1.0000    0.0000    1.0000
base = 'x*y'
       'z*y'
       'z'
sohs = 'x*y-z'
       '2*z*y'
       '7.21e-006*z'
g = y*x*x*y-y*x*z+4*y*z*z*y-z*x*y+1.00000000051984*z*z
```

- **example:**

```
>> [IsSohs,X,base,sohs,g]=NCsos(f,1e-4)
*************** Polynomial is SOHS ************
IsSohs = 1
X = 1.0000   -0.0000   -1.0000
   -0.0000    4.0000    0.0000
   -1.0000    0.0000    1.0000
base = 'x*y'
       'z*y'
       'z'
sohs = 'x*y-z'
       '2*z*y'
g = y*x*x*y-y*x*z+4*y*z*z*y-z*x*y+z*z
```

NCmin(f, precision)

- **description:**

  `[epsilon,X,base,sohs] = NCmin(f,precision)`
  computes the maximal ε such that the polynomial $f - \varepsilon$ is SOHS.

  example:

```
>> [epsilon,X,base,sohs]=NCmin(2*x+x^2+2*y+y^2)
epsilon = -2.0000
```

- **Demonstration:**

$$f = (1 + 3x_1^2 x2)^*(1 + 3x_1^2 x_2) - (x_1 x_2 + x_2^2)^*(x_1 x_2 + x_2^2)$$

Conclusions

- NCSOStooles in non-commutative extension for the commutative version SOStools.
- It is free-ware package to handle non-commutative polynomials.
- Newton chip method is a non-commutative version of the Newton polytope method.

Future tasks

- Optimize the preprocessing.
- Rounding scheme for rational solutions.
- Handling non-symmetric variables.

Forthcoming paper:

J. Povh, I. Klep: Semidefinite programming and sums of hermitian squares of noncommutative polynomials (submitted, 2008).